- valid sample space : 包含事件可能发生的每个情况.　　不遗漏. 不重复.

  countable finite (sample space) : 有无数个. 但能列举出来

- n中选k个
  - with replacement　n^k
  - without replacement
    - order matters　$n^{(k)}$
    - order doesn't matter　$\binom{n}{k}$

De Morgan's Law　$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|\overline{A})\,P(\overline{A}) + P(B|A)\cdot P(A)}$$

bayer's theorem

$$P(\overline{A}|B) = 1 - P(A|B)$$

$$P(A \cap B) = P(A|B)\,P(B) = P(B|A)\,P(A)$$

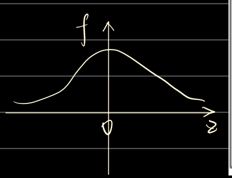
indep　$P(A|B) = P(A)$

mutually exclusive　$A \cap B = \emptyset$

→ behaviour : disjoint　$P(A_1 \cap A_2 | B) = P(A_1|B) + P(A_2|B)$

$$\begin{cases} \text{odds in favour 胜率} & \dfrac{P(A)}{1-P(A)} \\ \text{odds against 赔率} & \dfrac{1-P(A)}{P(A)} \end{cases}$$

• distribution	**Notation and Parameters**	**Probability Function** $f(x)$	**Mean** $E(X)$	**Variance** $Var(X)$
equally likely	Discrete Uniform(a,b) $b \geq a$ a,b integers	$\frac{1}{b-a+1}$ $x = a, a+1, \dots, b$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
不放回 N大 n小	Hypergeometric(N,r,n) $N = 1,2,\dots$ $n = 0,1,\dots,N$ $r = 0,1,\dots,N$	$\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $x = max(0, n-N+r),$ $\dots, min(r,n)$	$\frac{nr}{N}$	$\frac{nr}{N}\left(1-\frac{r}{N}\right)\frac{N-n}{N-1}$
	Binomial(n,p) $0 \leq p \leq 1, q = 1-p$ $n = 1,2,\dots$	$\binom{n}{x}p^x q^{n-x}$ $x = 0,1,\dots,n$	np	npq
n大p小	Bernoulli(p) $0 \leq p \leq 1, q = 1-p$	$p^x q^{1-x}$ $x = 0,1$	p	pq
X = 首次S前的#F.	Negative Binomial(k,p) $0 < p \leq 1, q = 1-p$ $k = 1,2,\dots$	$\binom{x+k-1}{x}p^k q^x$ $= \binom{-k}{x}p^k(-q)^x$ $x = 0,1,\dots$	$\frac{kq}{p}$	$\frac{kq}{p^2}$
X = k次S前的#F	Geometric(p) $0 < p \leq 1, q = 1-p$	pq^x $x = 0,1,\dots$	$\frac{q}{p}$	$\frac{q}{p^2}$
不涉及重叠性. 特定时间/空间内事件发生次数	Poisson(μ) $\mu \geq 0$	$\frac{e^{-\mu}\mu^x}{x!}$ $x = 0,1,\dots$	μ	μ
	Multinomial$(n; p_1, p_2, \dots, p_k)$ $0 \leq p_i \leq 1$ $i = 1,2,\dots,k$ and $\sum_{i=1}^{k} p_i = 1$	$f(x_1, x_2, \dots, x_k) =$ $\frac{n!}{x_1!\,x_2!\dots x_k!}p_1^{x_1}p_2^{x_2}\dots p_k^{x_k}$ $x_i = 0,1,\dots,n$ $i = 1,2,\dots,k$ and $\sum_{i=1}^{n} x_i = n$	$E(X_i)$ $= np_i$ $i = 1,\dots,k$	$Var(X_i)$ $= np_i(1-p_i)$ $i = 1,2,\dots,k$

	Uniform(a,b) $b > a$ (连续)	$\frac{1}{b-a}$ $a \leq x \leq b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
θ = waiting time / rate	Exponential(θ) $\theta > 0$	$\frac{1}{\theta}e^{-x/\theta}$ $x \geq 0$ 计算: gamma function: $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$	θ	θ^2
	(上述所有都可用N近似) $N(\mu,\sigma^2) = G(\mu,\sigma)$ $\mu \in \mathbb{R}, \sigma > 0$ $Z = \frac{X-\mu}{\sigma}$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$ $x \in \mathbb{R}$	μ	σ^2

- $\begin{cases} \text{Bi} & \text{结果由人为猜测决定} \\ \text{Po} & \text{客观. ep. 电话数/机械故障} \end{cases}$

$$\begin{cases} Po(\lambda) & f(x) = \dfrac{e^{-\mu} \cdot \mu^{x}}{x!} & F(x) = 1 - e^{-\lambda x} \\ Exp(\theta) & f(x) = \dfrac{1}{\theta} e^{-\frac{x}{\theta}} & F(x) = 1 - e^{-\frac{x}{\theta}} \end{cases}$$

- Random Variable

	discrete	cts.
pdf.	$P(X = x)$	$F'(x)$
cdf.	$P(X \in A) = \sum_{x \in X(S) \cap A} f(x)$	$P(a \le X \le b) = \int_a^b f(x) dx.$
$E(X)$	$\sum_{x \in X(S)} x f(x)$	$\int_{\mathbb{R}} x f(x) dx$
$Var(X)$	$E(X^2) - [E(X)]^2$	

$$\boxed{\begin{aligned} & E(aX+b) = aE(X) + b \\ & Var(aX+b) = a^2 \, Var(X) \\ & s.d.: \ \sigma = \sqrt{Var(X)} \end{aligned}}$$

$D \cap supp(f)$：求 f 在 domain D 中的积分

memoryless cts. distribution： $P(X > s+t \mid X > s) = P(X > t)$

☆ 用 $Y = f(x)$ 在 pdf. 中替换 X.

① 用 $F_X(y)$ 表示 $F_Y(y)$

② $pdf_X \to cdf_X \to cdf_Y \to pdf_Y$

③ f_Y in Range

– Multivariate

joint p.f.

$$\sum_{\text{all }(x,y)} f(x,y) = 1$$

		x			
	$f(x,y)$	0	1	2	$f_2(y) = P(Y=y)$
y	0	0.15	0.1	0.05	0.3
	1	0.35	0.2	0.15	0.7
	$f_1(x) = P(X=x)$	0.5	0.3	0.2	1

marginal p.f.

$$f(x_1, x_2, \cdots, x_k) = \frac{n!}{x_1! x_2! \cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

conditional p.f :

X given $Y=y$ $\quad f_1(x|y) = \dfrac{f(x,y)}{f_2(y)}$ $\quad$ Y given $X=x$ $\quad f_2(y|x) = \dfrac{f(x,y)}{f_1(x)}$

- X & Y independent,

$X \sim Po(\mu_1) \quad Y \sim Po(\mu_2) \qquad T = X+Y \sim Po(\mu_1+\mu_2)$

$X \sim Bi(n,p) \quad Y \sim Bi(m,p) \qquad T = X+Y \sim Bi(n+m, p)$

Expectation: $E[ag_1(X,Y) + bg_2(X,Y)] = aE[g_1(X,Y)] + bE[g_2(X,Y)]$

Variance : $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$

covariance : $Cov(X,Y) = \sigma_{XY} = E(XY) - E(X)E(Y)$ $\qquad xyf(x,y)$

$Cov(aX+bY, cU+dV) = ac\, Cov(X,U) + ad\, Cov(X,V) + bc\, Cov(Y,U) + bd\, Cov(Y,V)$

indep $\Leftrightarrow cov(X,Y) = 0 \Leftrightarrow E[g_1(X)g_2(Y)] = E[g_1(X)] \cdot E[g_2(Y)]$

correlation coefficient : $\rho = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

m.g.f : $M_X(t) = E(e^{tX})$ $\begin{cases} \text{discrete} & \sum_{x \in range(X)} e^{tx} f(x) \\ \text{continuous} & \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \end{cases}$

- $E(X^k) = M^{(k)}(0)$
- $M_X(t) = M_Y(t) \Rightarrow X$ & Y have the same distribution (unique theo for m.g.f)

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

4. James and Melanie have made a bet. They will take turns flipping a coin until a head is tossed. Whoever tosses the first head will win the bet and won't have to clean the bathrooms for the next year! James has elected to go first.

What are the respective probabilities that James will win and Melanie will win (in that order).

$$P(J\text{ win}) = P(\text{win in 1st}) + P(\text{win in 3rd}) + \cdots$$
$$= \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 \cdot \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right) + \cdots$$
$$= \tfrac{1}{2} \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{2n}$$
$$= \tfrac{1}{2} \times \frac{1}{1-\left(\frac{1}{2}\right)^2}$$
$$= \tfrac{2}{3}$$

$$P(M\text{ win}) = 1 - \tfrac{2}{3} = \tfrac{1}{3}$$

11. Jeremy has 8 different toy cars and 4 different toy trucks. He is hoping to take some vehicles on a holiday.

a. In how many ways can he select some of the vehicles to take on a holiday if he is to take all of the cars? ***(2 marks)***

$\frac{0}{1}$ _ _ _ $\quad 2^4 = 16$

b. In how many ways can he select some of the vehicles to take on a holiday if he is to take at least one car and brings no trucks with him? ***(2 marks)***

\# all choices − take no cars $\quad$ 8个空

$= 2^8 - 1$

$= 255$

c. In how many ways can he select some of the vehicles to take on a holiday if he is to take exactly 7 vehicles? ***(2 marks)***

$\binom{12}{7} = 792$

d. In how many ways can he select some of the vehicles to take on a holiday if he is to take exactly 3 cars and at least 2 trucks? ***(3 marks)***

$$\binom{8}{3}\binom{4}{2} + \binom{8}{3}\binom{4}{3} + \binom{8}{3}\binom{4}{4} = 616$$

Example: Suppose we make a random arrangement of length 3 using letters from the set $\{a, b, c, d, e, f, g, h, i, j\}$. What is the probability of the event B = "letters are in alphabetic order" if

(a) letters are selected without replacement?

(b) letters are selected with replacement?

(a) $$\frac{\binom{10}{3}}{10^{(3)}}$$

$\binom{10}{3}$: abc. acb. bac. bca. cba. cab

算作一种

(b) $N(3个重复) = 10$

$N(2个重复) = \binom{2}{1} \times 10^{(2)}$

$N(无重复) = \binom{10}{3}$

$$\frac{10 + \binom{2}{1} \times 10^{(2)} \times \binom{10}{3}}{10^{(3)}}$$

字母排序：1. 先选位置再选字母

2. 按是否有字母重复分类

3. Five tourists plan to attend Octoberfest. Each tourist attends a location selected at random from the choices: Alpine Club, Bingemans, Concordia Club, Kitchener Memorial Auditorium, Queensmount, Schwaben Club, Transylvania Club (7 locations). Find the probability that

(c) two tourists attend one location and three tourists attend another same location

_ _ _ _ _

选择2个位置，剩下3个位置一样.

前2位

后3位

$$\frac{\binom{5}{2} \times 7 \times 6}{7^5}$$

Markov Chains

transition probability matrix: $P_{ij} = P(X_{n+1} = j \mid X_n = i)$

$$P(X_1 = j) = \sum_{i=1}^{N} P_{ij} q_i$$

$$\underline{\pi^T} P = \underline{\pi^T}$$

每个事件的长期概率 $\pi_i = \lim_{t \to \infty} P(X_i = j)$